Numerical calculation of fully developed laminar flow in irregular annuli

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Received December 2003
Revised September 2005
Accepted September 2005

Abstract

Purpose – This study seeks to focus on the annular flow between rectangular and equilateral-triangular ducts under all possible arrangements. The aim of this work is to obtain accurate prediction of the friction factor of this flow using high-order finite element method.

Design/methodology/approach – Steady and fully developed laminar flow of incompressible Newtonian fluid in an annulus of variable cross-sectional geometry is investigated numerically. Accurate prediction of the friction factor of this flow was obtained using high-order finite element method.

Findings – The results were in agreement with already published findings in the literature. It was found that a higher annular area ratio will lead to a monotonic increase in *fRe* value in the case of regular annuli, and will lead to an increase followed by a decrease in *fRe* value in the case of irregular annuli. Also, it was, found that irregular annuli have lower *fRe* value than regular annuli, and that the square-in-triangle case has the lowest *fRe* value, whereas the square-in-square case has the highest *fRe* value.

Originality/value – Accurate prediction of the friction factor of the laminar flow in irregular annuli was obtained. Also, the obtained results can be utilized to optimize the annular geometries under consideration. In addition, the obtained results can lead to the design of more efficient heat exchangers.

Keywords Numerical analysis, Laminar flow, Friction

Paper type Technical paper

Nomenclature

A = cross-sectional area of flow
a = dimensional side-length of internal duct
b = dimensional side-length of external duct
AR = area ratio (inner duct to outer duct area ratio)
= $(a/b)^2 \times 100$ percent (SS-case)
= $(\sqrt{3}/4)(a/b)^2 \times 100$ percent (ST-case)
= $(4/\sqrt{3})(a/b)^2 \times 100$ percent (TS-case)
= $(a/b)^2 \times 100$ percent (TT-case)
D_h = hydraulic diameter, m
= $(b - a)$ (SS-case)
= $(4b^2 - \sqrt{3}a^2)/(4b + 3a)$ (ST-case)
= $(\sqrt{3}b^2 - 4a^2)/(3b + 4a)$ (TS-case)
= $(b - a)/\sqrt{3}$ (TT-case)
f = Fanning friction factor
m = degree of the interpolating polynomial
N = non-dimensional inward drawn normal
n = inward drawn normal

International Journal of Numerical Methods for Heat & Fluid Flow
Vol. 16 No. 3, 2006
pp. 293-303

0961-5539
DOI 10.1108/09615530610649726

P	= non-dimensional pressure	(X, Y, Z)	= non-dimensional Cartesian coordinates
p	= pressure, Pa	(x, y, z)	= Cartesian coordinates, m
Re	= Reynolds number		
(U, V, W)	= dimensionless velocities	*Greek symbols*	
(u, v, w)	= velocity components in x, y and z directions, respectively, m/s	ρ	= density of fluid, kg/m^3
$\overline{W}$	= non-dimensional average velocity	μ	= dynamic viscosity of fluid, kg/(m s)
$\bar{w}$	= average velocity of fluid, m/s	ν	= kinematic viscosity of fluid, m^2/s

Introduction

The flow in annuli of regular and irregular cross-sectional geometries is used in a wide range of industrial applications including, but not limited to, chemical, petroleum, pharmaceutical, food, and plastic industries. Therefore, it is practically important to be able to predict the pressure drop characteristics of the flow in a variety of annular geometries. In the past, diverse approximate techniques have been used to address this problem and to obtain the friction factor of the flow in various ducts and annular shapes. It should be emphasized here that if highly accurate results of friction factor are required, the commonly used approximate method, which uses the results of friction factor of regular annuli for irregular annuli of the same hydraulic diameter, is not sufficiently accurate for research and/or design purposes. Rather, one should re-solve the complete set of governing equations of the flow for that particular configuration under study.

The published research in this field can be divided into three categories:

(1) The flow in a single duct (e.g. cylinder, square, triangular duct, etc.).

(2) The flow in regular annuli (e.g. cylinder-in-cylinder, square-in-square ducts, etc.).

(3) The flow in irregular annuli (cylinder-in-square, square-in-cylinder ducts, etc.).

Within the first category, the fully developed laminar flow in a single duct of variable geometrical shapes has been considered by many researchers during the 1960s, 1970s and early 1980s. For a comprehensive review, one may refer to the reference book edited by Kakac *et al.* (1987). Furthermore, Nonino *et al.* (1988) solved the full Navier-Stokes equations for laminar three-dimensional parabolic flow in square ducts using the finite element method. Hydrodynamically developing flows in square ducts were presented. Their solutions were based on the parabolized simplification of the Navier-Stokes equations. New results were presented for developing flows in square ducts. Asaba *et al.* (1991) studied numerically the laminar flow in irregular domains. An algebraic coordinate transformation was developed to map the irregular domain onto a circle. Their results were comparable to existing and published data in the literature. Uzun and Unsal (1997) studied numerically the hydrodynamically fully developed laminar flow inside ducts of irregular cross-sections (e.g. triangular, sinusoidal, square ducts with truncated four corners, four cusped ducts and rhombic ducts). The elliptic grid generation technique was used to transform partial differential equations of arbitrary and irregular physical plane into a square shaped computational domain.

Among the researchers of the second category, Kuehn and Goldstein (1976) comprehensively studied the laminar flow in the annular region between horizontal concentric cylinders using experimental and theoretical approaches. The experimental and numerical studies of the eccentric case have been carried out by Kuehn and Goldstein (1978) and more recently by Guj and Stella (1995). El-Shaarawi and Alkam (1992) have solved the unsteady boundary layer equations using the finite difference technique to obtain the velocity profiles in the entrance region of a circular concentric annulus.

Comparatively, little work has been conducted in more complex flow domains, such as the irregular annuli considered in this study. Ratkowsky and Epstein (1968) applied the method of least square fitting of harmonic functions to known boundary conditions in order to obtain closed-form solutions for laminar flow of an incompressible fluid through the constant area annulus between a regular polygonal duct and a centered circular core. The product of Fanning friction factor and Reynolds number (i.e. fRe) was determined for the extreme cases of either no core or core touched the outer walls. Also the effect of polygonal corner angles on the numerical results was studied. Specifically, the following cases were under investigation: flow in triangular, octagonal, hexagonal and squared ducts all centered by circular cores. The above study by Ratkowsky and Epstein has actually motivated the present study.

It can be noticed that most researchers have focused their work on single tube of a particular cross-sectional geometry, and very few have addressed the irregular annuli. In this study, the different combinations of square and equilateral-triangular ducts will be considered, namely square duct in triangular duct, triangular duct in square duct, square duct in square duct and triangular duct in triangular duct. These cases will be studied for the full range of area ratio (AR) from zero (when there is no inner core) to the maximum possible value (when the inner duct touches the walls of the outer duct).

Obtaining highly accurate values of the friction factor for the fully developed laminar flow in irregular annuli was critical to our study. This was achieved by solving the appropriate governing equations and associated boundary conditions numerically using the finite element method. Our results should be utilized to optimize the annular geometries under consideration.

Mathematical formulation

Figure 1 shows a schematic diagram of the flow problem at hand. The flow is considered fully developed in the axial (z) direction. In this case, the laminar, viscous and incompressible flow of a constant property Newtonian fluid is governed by the following axial momentum equation:

$$-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0 \tag{1}$$

where w is the velocity component in the axial direction. The appropriate boundary conditions associated with the above governing equation are:

$$w=0 \quad \text{at the walls} \quad \frac{\partial w}{\partial n}=0 \quad \text{at the line(s) of symmetry} \tag{2}$$

where n is the inward-drawn normal.

For convenience in the subsequent analysis, equation (1) with its associated boundary conditions will be converted into non-dimensional form by introducing the following dimensionless variables:

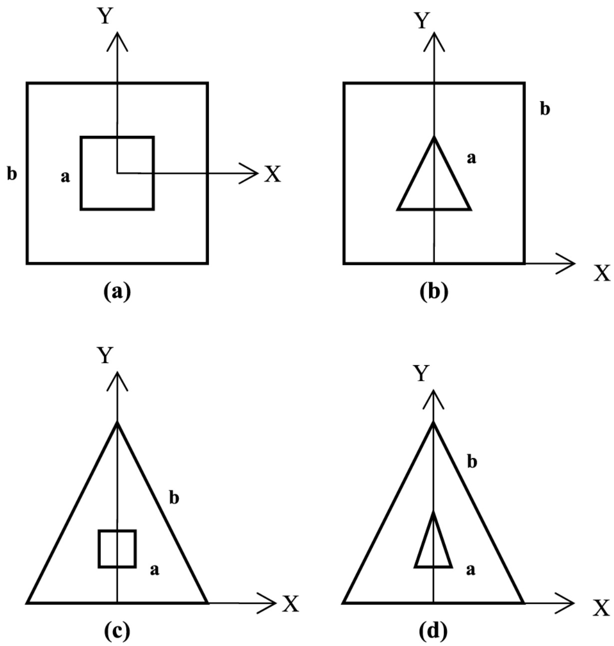

Figure 1.
Schematic diagram of the physical problem: (a) SS-case, (b) ST-case, (c) TS-case, (d) TT-case

$$X = \frac{x}{D_h}, \quad Y = \frac{y}{D_h}, \quad W = \frac{w}{\left(\frac{-1}{\rho}\frac{dp}{dz}\right)\left(D_h^2/\nu\right)} \tag{3}$$

Thus, the governing equation in non-dimensional form becomes:

$$\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right) + 1 = 0 \tag{4}$$

and is subjected to the following boundary conditions:

$$W = 0 \quad \text{at the walls} \quad \frac{\partial W}{\partial N} = 0 \quad \text{at the line(s) of symmetry} \tag{5}$$

The quantity of primary interest in the current study is the peripherally averaged Fanning friction factor of the irregular flow geometries at hand. This can be determined once the velocity field has been computed. The usual definition for this quantity is:

$$f = \frac{\bar{\tau}_w}{1/2(\rho \overline{W}^2)} \tag{6}$$

where $\bar{\tau}_w$ is the peripherally averaged wall shear stress. The product of the friction factor-Reynolds number (fRe) is expressed by considering the averaged velocity gradients on the duct walls:

$$fRe = \frac{-2}{\overline{W}}\overline{\left(\frac{\partial W}{\partial N}\right)_{\mathrm{w}}} \tag{7}$$

where f is the Fanning friction factor and Re is the Reynolds number, which is defined as:

$$Re = \frac{\rho \overline{W} D_{\mathrm{h}}}{\mu} \tag{8}$$

and the average velocity is defined as:

$$\overline{W} = \frac{1}{A}\iint_A W(X, Y)\mathrm{d}X\,\mathrm{d}Y \tag{9}$$

Numerical method of solution

The above set of governing partial differential equation and boundary conditions has been solved using the finite element technique. The domain of interest is divided into triangular elements of quadratic and cubic type, i.e. the degree of polynomial used to approximate the unknown field variables within each triangular element can be 2 or 3. The governing partial differential equation and boundary conditions are discretized through the well known Galerkin's formulation (Fletcher, 1984), and the resulting set of algebraic equations is solved iteratively by means of the Gauss-Seidel method. The details of the finite element method based on the Galerkin's formulation are available in many standard text books (Reddy, 1993). Solutions were assumed to converge when the following convergence criterion is satisfied by every dependent variable at every grid point in the computational domain:

$$\left|\frac{\Phi^{\mathrm{new}} - \Phi^{\mathrm{old}}}{\Phi^{\mathrm{new}}}\right| \leq 10^{-4} \tag{10}$$

where Φ could be any dependent variable.

Once the solution for dimensionless axial velocity field $W(X, Y)$ in a fixed flow geometry was obtained, the integral in equation (9) was calculated numerically, and subsequently the values of fRe were calculated using equation (7).

Results and discussion

Our study focuses on the friction factor of laminar flow in irregular annuli. For all cases considered, the flow cross-sectional area was modified by changing the annulus AR. The product of friction factor and Reynolds number (fRe) was tabulated and, based on these tables, optimum flow geometry was reached.

Grid sensitivity

Several numerical tests were carried out in order to verify the performance of the solution procedure. Laminar fluid flow in a single square duct was considered because the exact analytical expression of the friction factor-Reynolds number (fRe) product was available in the literature (Shah and Bhatti, 1987). For this flow problem, the exact value is $fRe = 14.227077$.

The numerical tests were performed using two different types of elements: quadratic ($m = 2$) and cubic ($m = 3$) triangular elements. The results were verified using a mesh refinement approach, i.e. computations were performed for several successively refined uniform meshes. The fRe values obtained for the different meshes used in our computation are shown in Table I. Here, it should be emphasized that the accuracy of the solution was improved by using higher-order elements. When compared with the above exact value, our results show an excellent agreement. This agreement has strengthened our confidence in the results and enabled us to move forward to the case of irregular annuli. The aforementioned mesh refinement approach was repeated for all cases considered, and the majority of the calculations presented here were made using third-order triangular elements. The total number of elements and nodes used were 508 and 1,077, respectively.

Irregular annuli

There were four cases considered in our study, namely: the square-in-square (SS-case), triangle-in-square (ST-case), square-in-triangle (TS-case), and triangle-in-triangle (TT-case). The triangle used in our study was equilateral. The numerical calculations were carried out for multiple annulus ARs.

Velocity contours

To study the effect of AR on the hydrodynamics of the flow in a given annular region, different values of the AR were considered for each case studied. Owing to geometric limitations (e.g. the walls of the inner duct cannot cross the walls of the outer duct), the values of the AR were not the same for all cases.

The following values of AR were tested for the SS-case: AR = 0, 10, 20, ..., 100 percent. For the case when AR = 0 (single square duct), the velocity was minimum at the walls and maximum at the center of the square, as expected. This case was used to validate our results.

Since the velocity contours for the different ARs were similar, and in order to save space, a sample velocity contour plot at AR = 20 percent will be shown for each annular geometry considered. The velocity contours of the SS-case are shown in Figure 2. By inspecting the figure, one can notice that the maximum velocity in the annular region occurs at 45° angle (measured from the horizontal axis with the origin

m	Number of elements	Number of nodes	fRe	m	Number of elements	Number of nodes	fRe
2	4	13	14.45814	2	764	1,591	14.22708
2	8	25	14.29393	2	900	1,875	14.22708
2	24	61	14.23519	3	4	21	14.35003
2	38	93	14.22950	3	16	72	14.23024
2	102	229	14.22743	3	34	138	14.22812
2	152	335	14.22726	3	120	460	14.22713
2	198	431	14.22717	3	202	756	14.22710
2	258	551	14.22713	3	330	1,228	14.22709
2	444	941	14.22710	3	338	1,434	14.22708
2	486	1,033	14.22709	3	402	1,483	14.22708
2	742	1,551	14.22708	3	432	1,591	14.22708

Table I.
Calculated values of fRe for the test case of a square duct

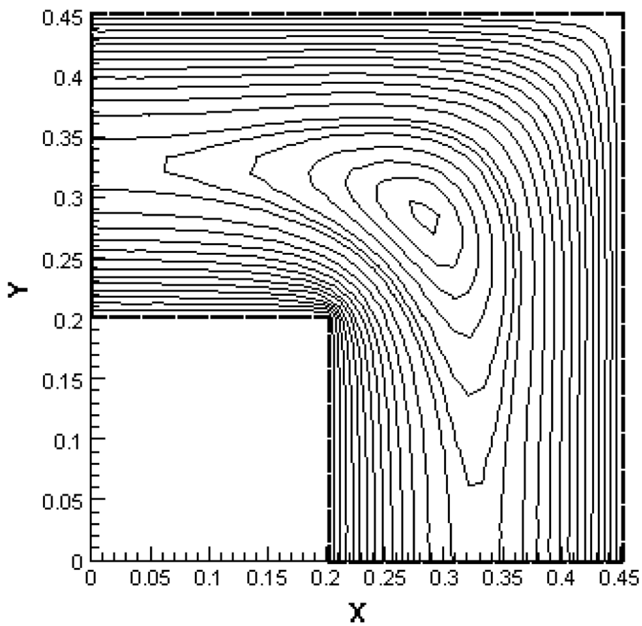

Figure 2.
SS-case: axial-velocity contours, AR = 20 percent

being at the geometric center of the squares). This is due to the fact that, at this angle, the space between the inner and outer walls of the annulus is at its maximum. Furthermore, the location of the maximum velocity lies not in the middle between the inner and outer walls but rather is closer to the inner wall.

The contour plot shown in Figure 2 was obtained again for the TS-case, and the result is shown in Figure 3. In this figure, it can be noted that the maximum velocity in the annular region occurs at 90° angle (measured from the horizontal) and is closer to the inner wall. This is fundamentally similar to the case shown in Figure 2, the reason being, the maximum velocity is expected to be located at an angle where, in general, the spacing between the inner and outer walls of the annulus is at its maximum.

The corresponding results were obtained for the rest of the cases considered. These are shown in Figure 4 for the case of ST-case, and in Figure 5 for the case of TT-case. The general trend discussed in Figures 2 and 3 above was also noted in these two cases, thus the discussion for these two cases will not be repeated here for clarity.

Effect of the annulus AR

Based on the above results for the velocity contours, the product of friction factor and Reynolds number (fRe) was calculated for all annular geometries considered in our study. The results are summarized in Table II. When these results are read carefully, one can make the following observations:

- *For all cases studied.* When AR = 0 (i.e. flow in single duct), the present study accurately predicts fRe value. This accuracy can be verified by comparing our results with the corresponding results in the literature (Ratkowsky and Epstein, 1968) which is shown in Table II.

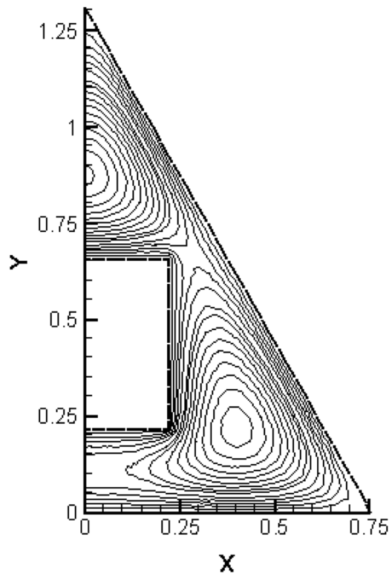

Figure 3.
ST-case: axial-velocity contours, AR = 20 percent

- *For regular annuli (SS-case and TT-case).* Any increase in AR will lead to a monotonic increase in *fRe* value. This trend was identical with that obtained by Ratkowsky and Epstein (1968) for the cylinder-in-cylinder regular annulus case.
- *For irregular annuli (ST-case and TS-case).* An increase in AR will lead to an initial increase followed by a decrease in *fRe* value. AR = 10 is a critical value in Table II at which point *fRe* is maximum for both ST-case and TS-case. This trend was also noted by Ratkowsky and Epstein (1968) for their irregular annuli.
- *For regular annuli.* When AR value is very large, the *fRe* value becomes insensitive to further increase in AR value. For example, *fRe* value becomes invariant when AR $\geq$ 90 percent for the SS-case, and when AR $\geq$ 80 percent for the TT-case.
- *For regular annuli.* As AR increases, *fRe* asymptotes to the same value. The asymptotic value in Table II is *fRe* = 24.00000 for both SS-case and TT-case. Ratkowsky and Epstein (1968) have obtained this same value in their study (Figure 2) for the cylinder-in-cylinder regular annulus case. This lead us to conclude that regardless of the particular shape used to generate a regular annulus (e.g. cylinder-in-cylinder or square-in-square or triangle-in-triangle, etc.), the asymptotic value of *fRe* will always be the same for all cases as AR approaches 100 percent.
- *For irregular annuli.* As AR value approaches extreme limits, i.e. when the wall of the inner duct becomes in contact with the wall of the outer duct, the *fRe* value of the TS-case asymptotes to a smaller value than that of the ST-case.
- *When AR is constant.* We can rank the geometric cases studied from smallest *fRe* value to largest in the following order: first TS-case then TT-case then ST-case then SS-case provided that AR = 10. When AR $>$ 10, the order becomes: TS-case then ST-case then TT-case and last SS-case.

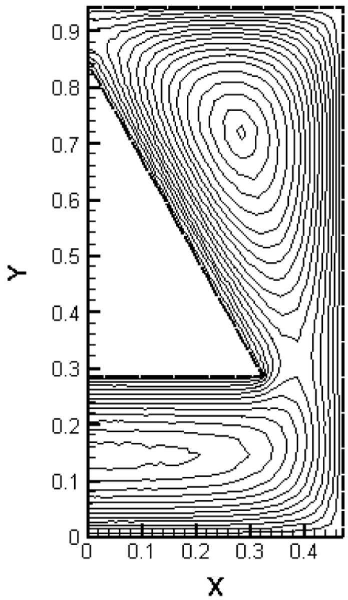

Figure 4. TS-case: axial-velocity contours, AR = 20 percent

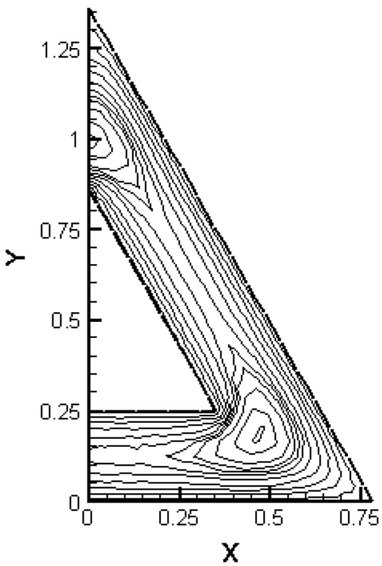

Figure 5. TT-case: axial-velocity contours, AR = 20 percent

Conclusions

The fully developed laminar flow of an incompressible Newtonian fluid in different irregular annuli was investigated by using the finite element technique. The annular configurations studied were made up of equilateral-triangular ducts and square ducts under all possible arrangements. Conclusions drawn from this study can be summarized by the following points:

fRe	AR percent	Case
14.22747 (14.23)[a]	0	
21.57767	10	
22.19143	20	
22.60156	30	
22.92255	40	SS-case
23.17198	50	
23.39839	60	
23.59690	70	
23.78603	80	
24.00000	90-100	
14.22747 (14.23)[a]	0	
21.28095	10	
20.19225	20	ST-case
17.74084	30	
17.15367	32	
13.33346 (13.33)[a]	0	
19.70767	10	
18.77559	20	TS-case
16.74992	30	
13.89206	40	
13.57214	41	
13.3336 (13.33)[a]	0	
20.55956	10	
21.43128	20	
22.00568	30	
22.50544	40	TT-case
22.86745	50	
23.23222	60	
23.55582	70	
24.00000	80-100	

Source: [a]Ratkowsky and Epstein (1968)

Table II.
Effect of AR on *fRe*

- *For regular annuli.* An increase in AR leads to asymptotic increase in the *fRe* value. The asymptotic value is independent on the shape of the annulus.
- *For irregular annuli.* An increase in AR will lead to an initial increase followed by a decrease in *fRe* value. AR = 10 was a critical value.
- *For regular annuli. fRe* value becomes insensitive to AR when AR approaches 100 percent.
- From a hydrodynamic point of view, the annular geometries studied can be ranked from best (i.e. lowest friction) to worst (i.e. largest friction) case as follows: TS-case, ST-case, TT-case and SS-case, respectively. One exception to this conclusion is when the AR is small, i.e. AR = 10, the order in this case becomes: TS-case, TT-case, ST-case and SS-case.
- *For all cases studied.* We found that the TS-case has the lowest friction loss whereas the SS-case has the highest friction loss (i.e. *fRe* value).

- In general, the results suggest the use of irregular annuli (triangular and square duct combinations) rather than the regular annuli (square-in-square and triangle-in-triangle duct combinations). Although irregular annuli are uncommonly used in industry at the present time, our study suggests that any improvement in the future on the pressure drop in double pipe flow devices must consider the use of dissimilar inner and outer duct shapes.

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